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ONE-PARAMETER GROUPS OF ISOMETRIES ON HARDY SPACES OF THE TORUS: SPECTRAL THEORY(1)

BY

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ABSTRACT. The spectral theory of the infinitesimal generator of an arbitrary one-parameter group of isometries on H^p of the torus, $1 \leq p < \infty, p \neq 2$, is considered. In particular, the spectrum of the generator is determined.

0. **Introduction.** The space H^p $(1 \leq p < +\infty)$ of the torus C^2 is defined (see [8]) as the subspace of complex $L^p(C^2)$ consisting of those functions whose double Fourier coefficients vanish on the complement of P, where P is the "positive set" in the character group of C^2, $P = \{(m,n): n > 0\} \cup \{(m,0): m \geq 0\}$. Throughout what follows we shall use (z,w) $(z = e^{i\theta}, w = e^{i\psi})$ to denote the general point of C^2, Z^p denotes the closure in $L^p(C^2)$ of the polynomials in z, and we identify Z^p with $H^p(d\theta/2\pi)$.

The strongly continuous one-parameter groups $\{T_t\}$ of isometries on $H^p(C^2)$, $1 \leq p < \infty, p \neq 2$, were described in [3]. The present paper is devoted to the spectral analysis of their infinitesimal generators, and thereby completes the investigation initiated in [3].

It is known that $\{T_t\}$ leaves Z^p invariant (see (1.1) below), and so the study of $\{T_t\}$ and its infinitesimal generator A naturally splits into two cases according to whether $\{T_t|Z^p\}$ has a bounded generator (§1) or an unbounded generator (§2). In the former case there are a real constant ρ and a real-valued measurable function $\delta(\cdot)$ on the circle C such that

$$(0.1) \qquad (T_t f)(z,w) = e^{i\rho t} f(z, e^{i\delta(z)t} w) \quad \text{for } f \in H^p(C^2).$$

As shown in §1, in this case the spectrum of A, $\Lambda(A)$, is the closure of the set $i\{\rho + \bigcup_{n=0}^{\infty} n\,[\text{ess. range}\,(\delta)]\}$. The dependence on $\delta(\cdot)$ of the fine structure of $\Lambda(A)$ is examined in the process.

When $\{T_t|Z^p\}$ has an unbounded generator, it is shown in [3, Theorems (2.22) and 2.24)] (by cocycle methods reminiscent of [6, Lecture V]) that there

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are real constants ρ and δ (with ρ unique), a unique nontrivial one-parameter group $\{\phi_t\}$ of Möbius transformations of the disc D, and a unimodular measurable function $u(\cdot)$ on C such that

$$(T_tf)(z,w) = e^{i\rho t}[\phi_t'(z)]^{1/p} f(\phi_t(z), e^{i\delta t}u(\phi_t(z))\overline{u(z)}w), \qquad \text{for } f \in H^p(C^2). \tag{0.2}$$

The properties of the group $\{T_t\}$ depend on the nature of the set $\mathcal{S}$ of common fixed points, in the extended plane, of the group $\{\phi_t\}$. The set $\mathcal{S}$ is known to be: (i) a doubleton set of symmetric points with respect to C, (ii) a singleton subset of C, or (iii) a doubleton subset of C (see [2]), and correspondingly, we shall say that $\{\phi_t\}$ is of *type* (i), (ii), or (iii). If $\{\phi_t\}$ is of type (ii) or (iii), we show in Theorem (2.1) that $\Lambda(A)$ is the imaginary axis. If $\{\phi_t\}$ is of type (i), we show in Theorem (2.2) that $\Lambda(A)$ is either a bilateral arithmetic progression of pure-imaginary numbers, or the whole imaginary axis.

The following notation will be used throughout. The symbols $\mathbf{Z}$, $\mathbf{Z}^+$, $\mathbf{R}$, and $\mathbf{C}$ will denote, respectively, the set of all integers, the set of nonnegative integers, the real line, and the complex plane. Normalized Lebesgue measure on C (resp., on C^2) will be symbolized by μ (resp., $\tilde{\mu}$).

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1. **Spectral analysis when $\{T_t|Z^p\}$ has a bounded generator.** For convenience we record here the following theorem of Lal and Merrill [8]:

THEOREM. *If T is a linear isometry of $H^p(C^2)$ onto $H^p(C^2)$ $(1 \leqslant p < \infty, p \neq 2)$, then there are $\alpha \in C$, a Möbius transformation of the disc ϕ, and a measurable function σ: $C \to C$ such that for all $f \in H^p(C^2)$:*

$$(Tf)(z,w) = \alpha[\phi'(z)]^{1/p} f(\phi(z), \sigma(z)w). \tag{1.1}$$

On the other hand, if $1 \leqslant p < \infty$, and α, ϕ, and σ are as above, the right-hand side of (1.1) *defines a linear isometry of $H^p(C^2)$ onto $H^p(C^2)$.*

For $1 \leqslant p < \infty$, let $\mathcal{K}_p$ be the set of all strongly continuous one-parameter groups $\{T_t\}$ of isometries of $H^p(C^2)$ such that $\{T_t|Z^p\}$ has a bounded infinitesimal generator (if $p = 2$, we further require that each T_t have the form (1.1)). The following theorem describes $\mathcal{K}_p$ [3, Theorem (1.5)].

(1.2) THEOREM. *Let $\{T_t\} \in \mathcal{K}_p$, $1 \leqslant p < \infty$. Then there are a real constant ρ and a real-valued measurable function $\delta(\cdot)$ on C such that*

$$(T_tf)(z,w) = e^{i\rho t} f(z, e^{i\delta(z)t}w) \quad \text{for } t \in \mathbf{R}, f \in H^p(C^2). \tag{1.3}$$

Conversely, for any such ρ and $\delta(\cdot)$, (1.3) *defines a group $\{T_t\} \in \mathcal{K}_p$.*

REMARK. It is easy to see that for the group $\{T_t\}$ given by (1.3), ρ is uniquely determined, and $\delta(\cdot)$ is uniquely determined up to equality almost everywhere.

We also have the following description of the infinitesimal generator [3, Theorem (1.7)].

(1.4) THEOREM. *For* $1 \leqslant p < \infty$, *let* $\{T_t\}$ *in* $\mathcal{K}_p$ *have the form* (1.3), *and let* A *be the infinitesimal generator of* $\{T_t\}$. *Then the domain of* A, $\mathfrak{D}(A)$, *consists of those functions* $f \in H^p(C^2)$ *for which there is a* $g \in H^p(C^2)$ *so that for almost all* z *the following hold*:

(i) $\delta(z) = 0$ *implies* $g(z,w) = 0$ *for almost all* w;

(ii) $\delta(z) \neq 0$ *implies that there is a (necessarily unique) function* F_z *on* C *such that* $F_z(e^{i\psi})$ *is absolutely continuous for* $0 \leqslant \psi \leqslant 2\pi$, $f(z,w) = F_z(w)$ *for almost all* w, *and*

$$g(z,e^{i\psi}) = \delta(z)\frac{d}{d\psi}(F_z(e^{i\psi}))$$

for almost all ψ. *If* f *belongs to* $\mathfrak{D}(A)$ *and* g *is as above, then*

$$Af = i\rho f + g.$$

In order to describe the spectrum of A we shall need the following standard notion.

DEFINITION. If (X,ν) is a measure space, $\nu \geqslant 0$, and $h: X \to \mathbf{R}$ is a ν-measurable function, the *ν-essential range* of h (abbreviated ν-ess. range (h)) is the set of all $x \in \mathbf{R}$ such that for each open neighborhood N of x, $\nu(h^{-1}(N)) > 0$.

We now turn our attention to proving the following theorem.

(1.5) THEOREM. *Let* A *be the infinitesimal generator of a group* $\{T_t\}$ *in* $\mathcal{K}_p$, $1 \leqslant p < \infty$, *and let* δ *and* ρ *be as in* (1.3). *Then the spectrum of* A *is the closure of the set* $i\{\rho + \bigcup_{n=0}^{\infty} n[\mu\text{-ess. range } (\delta)]\}$.

Theorem (1.5) will follow from a succession of results established below, which also provide information about the fine structure of $\Lambda(A)$.

Since a change in ρ only translates the generator and its spectrum, we shall for convenience study the group $\{S_t\}$ given by

$$(S_t f)(z,w) = f(z, e^{i\delta(z)t}w).$$

In order to avoid interruption of the flow of ideas later on, we now introduce some terminology and state some convenient facts whose simple proofs will be omitted.

DEFINITION. Let ν be the product measure on $\mathbf{Z}^+ \times C$, where $\mathbf{Z}^+$ is endowed with discrete measure and C has Lebesgue measure μ. Let Δ: $\mathbf{Z}^+ \times C \to \mathbf{R}$ be the function defined by $\Delta(n, z) = n\delta(z)$. Denote by $\mathfrak{S}$ the ν-essential range of Δ.

(1.6) *For each* $\lambda \in \mathbf{R}$, $\lambda \in \mathfrak{S}$ *if and only if for each* $\varepsilon > 0$, *there are* $n_\varepsilon \in \mathbf{Z}^+$ *and* $M_\varepsilon \subseteqq C$ *such that* $\mu(M_\varepsilon) > 0$ *and* $|\lambda - n_\varepsilon \delta(z)| < \varepsilon$ *for* $z \in M_\varepsilon$. (*In particular* $0 \in \mathfrak{S}$.)

(1.7) $\mathfrak{S}$ *is the closure of* $\bigcup_{n=0}^{\infty} n[\mu\text{-ess. range } (\delta)]$.

(1.8) *For each* $\lambda \in \mathbf{R}$, λ *is an accumulation point of* $\mathfrak{S}$ *if and only if for each* $\varepsilon > 0$, *there are* $n_\varepsilon \in \mathbf{Z}^+$, $\eta_\varepsilon > 0$, *and* $M_\varepsilon \subseteqq C$ *such that* $\mu(M_\varepsilon) > 0$ *and* $\eta_\varepsilon \leqslant |\lambda - n_\varepsilon \delta(z)| < \varepsilon$ *for* $z \in M_\varepsilon$.

(1.9) *If for every* $\gamma > 0$ *the inverse-image* $\delta^{-1}\{(0, \gamma)\}$ (*resp.*, $\delta^{-1}\{(-\gamma, 0)\}$) *has positive measure, then* $\mathfrak{S}$ *contains every positive* (*resp.*, *every negative*) *real number.*

We shall require some further notation. For $f \in L^p(d\tilde{\mu})$ and $n \in \mathbf{Z}$, let

$$a_f^{(n)}(z) = \frac{1}{2\pi} \int_0^{2\pi} f(z, e^{i\psi}) e^{-in\psi} \, d\psi.$$

Clearly, $\|a_f^{(n)}\|_p \leqslant \|f\|_p$, where the p-norms are taken, respectively, in $L^p(d\theta/2\pi)$ and $L^p(d\tilde{\mu})$. The closed linear span in $L^p(d\tilde{\mu})$ of the functions $z^m w^n$, $m, n \in \mathbf{Z}$, $n \geqslant 1$, will be denoted by W^p. It is known [9, Lemma 5] that $H^p(C^2) = Z^p \oplus W^p$. It is easy to see that the projection on Z^p along W^p is given by $f \mapsto a_f^{(0)}$. Notice that if $f \in H^p(C^2)$, then for almost all $z \in C$, $a_f^{(-n)}(z) = 0$ for $n = 1, 2, \ldots$. Thus for almost all $z \in C, f(z, \cdot) \in H^p(d\psi/2\pi)$.

Returning now to the group $\{S_t\}$, we set $E_{n,\lambda} = \{z \in C\colon n\delta(z) = \lambda\}$ for $\lambda \in \mathbf{R}$, $\lambda \neq 0$, and $n = 1, 2, \ldots$. Then $\{E_{n,\lambda}\}_{n=1}^{\infty}$ is a sequence of disjoint measurable sets.

(1.10) Theorem. *Let* $\mathfrak{A}$ *be the infinitesimal generator of the group* $\{S_t\}$ *acting in* $H^p(C^2)$, $1 \leqslant p < \infty$, *by*

$$(S_t f)(z, w) = f(z, e^{i\delta(z)t} w).$$

Then for $\lambda \in \mathbf{R}$, $\lambda \neq 0$, *we have*:

(i) *The closure of the range of* $(\mathfrak{A} - i\lambda)$ *is the set* $\{g \in H^p(C^2)$: *for* $n = 1, 2, \ldots, a_g^{(n)}(z) = 0$ *for almost all* $z \in E_{n,\lambda}\}$.

(ii) *The null-space of* $(\mathfrak{A} - i\lambda)$ *is the set* $\{f \in W^p$: *for* $n = 1, 2, \ldots, a_f^{(n)}(z) = 0$ *for almost all* z *in* $C \backslash E_{n,\lambda}\}$.

(iii) *The space* $H^p(C^2)$ *is the direct sum of the closure of the range of* $(\mathfrak{A} - i\lambda)$ *and the null space of* $(\mathfrak{A} - i\lambda)$.

Proof. Let Γ be the set of all $g \in H^p(C^2)$ such that for $n = 1, 2, \ldots, a_g^{(n)}$ vanishes almost everywhere in $E_{n,\lambda}$. The fact that Γ is closed follows from the observation that for each n, $g \mapsto \chi_{E_{n,\lambda}} a_g^{(n)}$ (where χ denotes characteristic function) defines a bounded operator from $H^p(C^2)$ into $L^p(d\theta/2\pi)$. Now suppose $(\mathfrak{A} - i\lambda)f = g$. From (1.4)(ii) it follows that if $\delta(z) \neq 0$,

$$\delta(z)\frac{dF_z(e^{i\psi})}{d\psi} - i\lambda F_z(e^{i\psi}) = g(z, e^{i\psi}),$$

for almost all ψ. Taking Fourier coefficients we get

$$i(n\delta(z) - \lambda)a_f^{(n)}(z) = a_g^{(n)}(z).$$

It is now clear that Γ contains the range of $(\mathcal{Q} - i\lambda)$, and hence its closure.

To prove the reverse inclusion, we first observe that if n is any positive integer and $h \in L^p(d\theta/2\pi)$ vanishes almost everywhere on $E_{n,\lambda}$, then $h(z)w^n$ is in the closure of the range of $(\mathcal{Q} - i\lambda)$. In fact, for each positive integer k, let χ_k be the characteristic function of the set $\{z \in C: |n\delta(z) - \lambda| > 1/k\}$. Now define

$$f_k(z, w) = \left[\frac{\chi_k(z)}{i(n\delta(z) - \lambda)}\right]h(z)w^n.$$

It is easy to check that $(\mathcal{Q} - i\lambda)f_k = \chi_k(z)h(z)w^n$. It follows from the dominated convergence theorem that $h(z)w^n$ is in the closure of the range of $(\mathcal{Q} - i\lambda)$.

Let now $g \in \Gamma$. It is clear from the preceding observation that for each $n = 1, 2, \ldots, a_g^{(n)}(z)w^n$ is in the closure of the range of $(\mathcal{Q} - i\lambda)$. Also, it is obvious from (1.4) that $\mathcal{Q}a_g^{(0)} = 0$ and $a_g^{(0)}$ is in the range of $(\mathcal{Q} - i\lambda)$. Let $g_n(z, w)$ be the nth Cesàro mean of the series $\sum_{k=0}^{\infty} a_g^{(k)}(z)w^k$. By the foregoing, g_n belongs to the closure of the range of $(\mathcal{Q} - i\lambda)$. From standard facts about the convergence of Cesàro means $\{g_n\}$ converges to g in $H^p(C^2)$, and the proof of (1.10)(i) is complete.

Now we turn to the proof of (1.10)(ii). For any $f \in H^p(C^2)$, consider the series

$$\sum_{n=1}^{\infty} \chi E_{n,\lambda}(z)a_f^{(n)}(z)w^n. \tag{1.11}$$

Since the $E_{n,\lambda}$'s are disjoint, for each z this series has at most one term not identically zero in w. We have

$$\left|\sum_{n=1}^{k} \chi_{E_{n,\lambda}}(z)a_f^{(n)}(z)w^n\right| \leqslant \left[\frac{1}{2\pi}\int_0^{2\pi} |f(z, e^{i\psi})|^p\, d\psi\right]^{1/p}. \tag{1.12}$$

Let $h(z, w)$ denote the pointwise sum of the series in (1.11). It is clear from (1.12) that this series converges to h in $L^p(C^2)$, and therefore $h \in W^p$, with $\|h\|_p \leqslant \|f\|_p$. Let us define Q: $H^p(C^2) \to H^p(C^2)$ by setting $Qf = h$. From the foregoing Q is a well-defined idempotent linear map with $\|Q\| \leqslant 1$.

It is easy to see that the null space of Q consists of all $f \in H^p(C^2)$ such that for $n = 1, 2, \ldots, a_f^{(n)}(z) = 0$ for almost all z in $E_{n,\lambda}$. Thus by (1.10)(i) we have

(1.13) The null space of Q is the closure of the range of $(\mathfrak{C} - i\lambda)$.

Next we note that if $f \in H^p(C^2)$, then for almost all z, $a_{Qf}^{(n)}(z) = \chi_{E_{n,\lambda}}(z)a_f^{(n)}(z)$ for $n \geqslant 1$, $a_{Qf}^{(0)}(z) = 0$, and $a_{Qf}^{(n)}(z) = a_f^{(n)}(z) = 0$ for $n < 0$. From this we see that the range of Q consists of all $f \in W^p$ such that for $n = 1, 2, \ldots, a_f^{(n)}(z) = 0$ for almost all z in $C \backslash E_{n,\lambda}$. From the definition of Q and Theorem (1.4) we find that the range of Q is contained in the null space of $(\mathfrak{C} - i\lambda)$. Conversely, suppose $\mathfrak{C}f = i\lambda f$. It is easy to see from Theorem (1.4) that the range of $\mathfrak{C}$ is contained in W^p, and so $f \in W^p$. It follows readily from Theorem (1.4) that for almost all z, $(n\delta(z) - \lambda)a_f^{(n)}(z) = 0$ for every integer n. Thus f is in the range of Q, as characterized above. In summary we have:

(1.14) The null space of $(\mathfrak{C} - i\lambda)$ is the range of Q, and the range of Q consists of all $f \in W^p$ such that for $n = 1, 2, \ldots, a_f^{(n)}(z) = 0$ for almost all z in $C \backslash E_{n,\lambda}$.

Thus (1.10)(ii) is established, and, in view of (1.13), so is (1.10)(iii).

Now we take up the companion theorem (for the case $\lambda = 0$) to Theorem (1.10).

(1.15) THEOREM. *Let* $N(\delta) = \{z \in C\colon \delta(z) = 0\}$ *and let* $K(z,w) = \chi_{N(\delta)}(z)$. *Then with the same notation as in* (1.10), *we have*:

(i) *The closure of the range of* $\mathfrak{C}$ *is* $(1 - K)W^p$.

(ii) *The null-space of* $\mathfrak{C}$ *is* $Z^p + KW^p$.

(iii) *The space* $H^p(C^2)$ *is the direct sum of the closure of the range of* $\mathfrak{C}$ *and the null-space of* $\mathfrak{C}$.

PROOF. Let g be in the range of $\mathfrak{C}$. Then by Theorem (1.4), $(1 - K)g = g$ and as noted earlier $g \in W^p$. Obviously, $(1 - K)W^p$ is closed and so by the foregoing contains the closure of the range of $\mathfrak{C}$. To see the reverse inclusion, for $k = 1, 2, \ldots,$ let χ_k be the characteristic function of the set $\{z \in C\colon |\delta(z)| \geqslant k^{-1}\}$. Let m, n be integers with $n > 0$ and define

$$f_k(z,w) = \left[\frac{\chi_k(z)}{i\delta(z)}\right] z^m w^n.$$

In a fashion similar to the proof of (1.10)(i) it follows that $(1 - K)z^m w^n$ is in the closure of the range of $\mathfrak{C}$, and this completes the proof of (1.15)(i).

We now prove the less obvious inclusion in (1.15)(ii). Suppose $\mathfrak{C}f = 0$. Write $f = u + v$, $u \in Z^p$, $v \in W^p$. By Theorem (1.4), $\mathfrak{C}u = 0$, and hence for almost all $z \in C$ if $\delta(z) \neq 0$, $v(z,w)$ has the same value for almost all w. This constant value must be $a_v^{(0)}(z) = 0$ since $v \in W^p$. Hence $v = Kv$ and $f \in Z^p + KW^p$.

DEFINITION. For $\lambda \in \mathbf{R}$, we denote by Q_λ the projection of $H^p(C^2)$ on the null space of $(\mathcal{Q} - i\lambda)$ along the closure of the range of $(\mathcal{Q} - i\lambda)$.

The stage is now set for considering the fine structure of the spectrum of $\mathcal{Q}$.

(1.16) THEOREM. *The point spectrum of* $(-i\mathcal{Q})$ *is*

$$\{0\} \cup \left\{\lambda \in \mathbf{R}\Big\backslash\{0\}: \mu\left(\bigcup_{n=1}^{\infty} E_{n,\lambda}\right) > 0\right\}.$$

PROOF. Suppose $\lambda \in \mathbf{R}$, $\lambda \neq 0$, and for some positive integer j we have $\mu(E_{j,\lambda}) > 0$. Then by (1.10)(ii), the function $\chi_{E_{j,\lambda}}(z)w^j$ is in the null-space of $(\mathcal{Q} - i\lambda)$.

REMARK. Theorem (1.16) shows that the point spectrum of $(-i\mathcal{Q})$ is a countable union of arithmetic progressions. Specifically, the point spectrum of $(-i\mathcal{Q})$ is the union of $\{0\}$ and

$$\bigcup\{\alpha\mathbf{Z}^+: \alpha \in \mathbf{R}, \mu(\delta^{-1}(\alpha)) > 0\}.$$

The set $\mathfrak{S}$ discussed in (1.6)–(1.9) now enters the stage.

(1.17) THEOREM. *Let* $\lambda \in \mathbf{R}$. *The range of* $(\mathcal{Q} - i\lambda)$ *is closed if and only if* λ *is not an accumulation point of* $\mathfrak{S}$. *In this case, there is a bounded linear operator* $J_\lambda: H^p(C^2) \to H^p(C^2)$ *such that* $J_\lambda(H^p(C^2)) \subseteqq \mathfrak{D}(\mathcal{Q})$ *and* $(\mathcal{Q} - i\lambda)J_\lambda f = f - Q_\lambda f$ *for all* $f \in H^p(C^2)$.

PROOF. We first prove the existence of J_λ for $\lambda \in \mathbf{R}$, $\lambda \neq 0$, where λ is not an accumulation point of $\mathfrak{S}$. Let $B_\lambda^{(+)}$ (resp., $B_\lambda^{(-)}$) be the set of all $z \in C\backslash N(\delta)$ such that $\operatorname{sgn}\delta(z) = \operatorname{sgn}\lambda$ (resp., $\operatorname{sgn}\delta(z) = -\operatorname{sgn}\lambda$), where "sgn" stands for "sign of". For $n = 1, 2, \ldots,$ define G_n on C by $G_n(z) = 0$ if $z \in E_{n,\lambda} \cup N(\delta)$, and $G_n(z) = [\operatorname{sgn}\delta(z)](n\delta(z) - \lambda)^{-1}$ if $z \notin E_{n,\lambda} \cup N(\delta)$. Also, let $G_0(z) = -[\operatorname{sgn}\delta(z)]/\lambda$ for $z \in C\backslash N(\delta)$, and $G_0(z) = 0$ for $z \in N(\delta)$. Since λ is not an accumulation point of $\mathfrak{S}$, it follows from (1.8) and (1.9), respectively, that there are $\varepsilon_\lambda > 0$ and $\gamma_\lambda > 0$ such that

$$\|G_n\|_\infty \leqslant \varepsilon_\lambda^{-1} \quad \text{for } n = 1, 2, \ldots, \tag{1.18}$$

and

$$|\delta(z)| \geqslant \gamma_\lambda \quad \text{for almost all } z \in B_\lambda^{(+)}. \tag{1.19}$$

For $z \in B_\lambda^{(-)}$ (resp., $B_\lambda^{(+)}$) the sequence (resp., a tail of the sequence) $\{G_n(z)\}_{n=0}^\infty$ is convex. Thus by [5, Theorem (4.5)] for each $z \in C\backslash N(\delta)$ the series

$$G_0(z) + 2\sum_{n=1}^{\infty} G_n(z)\operatorname{Re}(w^n) \tag{1.20}$$

converges for all $w \in C\backslash\{1\}$. Since the series in (1.20) converges trivially to 0 for $z \in N(\delta)$, we define a function $L(z,w)$ on $C\chi(C\backslash\{1\})$ by (1.20). For $z \in B_\lambda^{(-)}$ we also have from [5, Theorem (4.5)] that

$$L(z,w) \geqslant 0 \quad \text{and} \quad (2\pi)^{-1}\int_0^{2\pi} L(z,e^{i\psi})e^{-in\psi}\,d\psi = G_n(z) \tag{1.21}$$
$$\text{for } n = 0, 1, 2, \ldots.$$

Let n_λ be the largest integer less than or equal to $|\lambda|\gamma_\lambda^{-1}$. It follows from (1.19) that for almost all $z \in B_\lambda^{(+)}|G_n(z)| \leqslant (n\gamma_\lambda - |\lambda|)^{-1}$ for $n > n_\lambda$. Thus by (1.18) we have for almost all $z \in B_\lambda^{(+)}$,

$$\sum_{n=1}^{\infty} |G_n(z)|^2 \leqslant n_\lambda \varepsilon_\lambda^{-2} + \sum_{n=n_\lambda+1}^{\infty} (n\gamma_\lambda - |\lambda|)^{-2}.$$

For such z in $B_\lambda^{(+)}$ we see from (1.20) that

$$\begin{aligned}(2\pi)^{-1}\int_0^{2\pi} |L(z,e^{i\psi})|^2\,d\psi &= |G_0(z)|^2 + 2\sum_{n=1}^{\infty}|G_n(z)|^2 \\ &\leqslant |\lambda|^{-2} + 2n_\lambda\varepsilon_\lambda^{-2} + 2\sum_{n=n_\lambda+1}^{\infty}(n\gamma_\lambda - |\lambda|)^{-2}.\end{aligned} \tag{1.22}$$

It is obvious from (1.21) that for $z \in B_\lambda^{(-)}$, $(2\pi)^{-1}\int_0^{2\pi}|L(z,e^{i\psi})|\,d\psi = |\lambda|^{-1}$. Let us denote the square root of the majorant in (1.22) by M_λ. Then it is immediate from the foregoing that:

(1.23) For almost all $z \in C$, $(2\pi)^{-1}\int_0^{2\pi}|L(z,e^{i\psi})|\,d\psi \leqslant M_\lambda$. Now define a function $k_\lambda(\cdot,\cdot)$ by setting $k_\lambda(z,w) = -i[\operatorname{sgn}\delta(z)]L(z,w)$ if $z \in C\backslash N(\delta)$, $k_\lambda(z,w) = 0$ if $z \in N(\delta)$. For each $f \in H^p(C^2)$ we see from (1.23) that the integral $\int_0^{2\pi} k_\lambda(z,we^{-iu})f(z,e^{iu})\,du$ exists for almost all $(z,w) \in C^2$ and defines a $\tilde{\mu}$-integrable function of (z,w). We set

$$\begin{aligned}(J_\lambda f)(z,w) &= (2\pi)^{-1}\int_0^{2\pi} k_\lambda(z,we^{-iu})f(z,e^{iu})\,du \\ &\quad + i\lambda^{-1}\chi_{N(\delta)}(z)f(z,w), \quad \text{for } f \in H^p(C^2).\end{aligned} \tag{1.24}$$

We first observe from (1.23) and (1.24) that for almost all $z \in C\backslash N(\delta)$:

$$\int_0^{2\pi} |(J_\lambda f)(z,e^{i\psi})|^p\,d\psi \leqslant M_\lambda^p \int_0^{2\pi}|f(z,e^{i\psi})|^p\,d\psi. \tag{1.25}$$

Since $M_\lambda \geqslant |\lambda|^{-1}$, it is clear that (1.25) holds for $z \in N(\delta)$. It follows that J_λ is a well-defined linear transformation from $H^p(C^2)$ into $L^p(\tilde{\mu})$ with $\|J_\lambda\| \leqslant M_\lambda$. If we denote $J_\lambda f$ by g, then it is immediate from (1.24) that for almost all $z \in C$, $a_g^{(n)}(z) = 0$ for $n < 0$, $a_g^{(0)}(z) = i\lambda^{-1}a_f^{(0)}(z)$. This shows that $g \in H^p(C^2)$. Moreover, for almost all $z \in C\backslash N(\delta)$

$$(1.26)\qquad a_g^{(n)}(z) = a_{k_\lambda}^{(n)}(z)a_f^{(n)}(z) \quad \text{for } n = 0, \pm 1, \pm 2, \ldots.$$

Let z_0 be an arbitrary point in $C\backslash N(\delta)$ such that (1.26) holds. Since the $E_{n,\lambda}$'s are disjoint, it is clear that for all sufficiently large n (say $n \geqslant n_0 > 0$), $z_0 \notin E_{n,\lambda}$, and so for $n \geqslant n_0$ (1.26) becomes

$$(1.27)\qquad a_g^{(n)}(z_0) = -i(n\delta(z_0) - \lambda)^{-1}a_f^{(n)}(z_0).$$

Consider now the function $h(w)$ (defined for $|w| < 1$) by $\sum_{n=n_0}^{\infty} a_g^{(n)}(z_0)w^n$. This function is in H^p of the disc, and we now show that dh/dw is also in H^p of the disc. Clearly by (1.27)

$$h'(w) = -i \sum_{n=n_0}^{\infty} n(n\delta(z_0) - \lambda)^{-1}a_f^{(n)}(z_0)w^{n-1}.$$

So

$$(1.28)\quad i\delta(z_0)h'(w) = \sum_{n=n_0}^{\infty} a_f^{(n)}(z_0)w^{n-1} + \lambda \sum_{n=n_0}^{\infty} (n\delta(z_0) - \lambda)^{-1}a_f^{(n)}(z_0)w^{n-1}.$$

The first series on the right of (1.28) is obviously in H^p. It follows from Hardy's inequality [5, p. 48] that $\sum_{n=n_0}^{\infty} |n\delta(z_0) - \lambda|^{-1}|a_f^{(n)}(z_0)| < \infty$. So the second series on the right of (1.28) is in H^∞. By [5, Theorem (3.11)], it now follows that $g(z_0, e^{i\psi})$ is equal for almost all ψ to an absolutely continuous function of ψ. From this it is readily verified (by taking Fourier coefficients with respect to ψ for each fixed z in a set of full measure) that by virtue of Theorem (1.4) $g \in \mathfrak{D}(\mathfrak{A})$ and $(\mathfrak{A} - i\lambda)g = f - Q_\lambda f$ in $H^p(C^2)$.

Now we prove the existence of J_0 with the stated properties under the assumption that 0 is not an accumulation point of $\mathfrak{S}$. For arbitrary f in $H^p(C^2)$, write $f = a_f^{(0)} + v$, where $v \in W^p$. Define V on C^2 by $V(z, e^{i\psi}) = \int_0^\psi v(z, e^{iu})\,du$. It is easy to see that $V \in L^p(\tilde{\mu})$, and $\|V\|_p \leqslant 2\pi\|v\|_p \leqslant 4\pi\|f\|_p$, and also that $(V(z,w) - a_V^{(0)}(z))$ is in W^p. Define $J_0 f$ on C^2 as follows: $(J_0 f)(z,w) = 0$ if $z \in N(\delta)$, and

$$(J_0 f)(z,w) = [\delta(z)]^{-1}(V(z,w) - a_V^{(0)}(z)) \quad \text{if } z \in C\backslash N(\delta).$$

Since 0 is not an accumulation point of $\mathfrak{S}$, we get from (1.9) that there is $\eta > 0$ such that $|\delta(z)| \geqslant \eta$ for almost all z in $C\backslash N(\delta)$. It follows that $J_0 f \in W^p$, and $\|J_0 f\|_p \leqslant (8\pi/\eta)\|f\|_p$. It is straightforward to verify that $\mathfrak{A}(J_0 f) = f - Q_0 f$. We remark in passing that the foregoing shows that $\|J_\lambda\|$ has a majorant which depends only on λ (and δ), but not on p.

Now we show that if $\lambda \in \mathbf{R}$ is not an accumulation point of $\mathfrak{S}$, then $(\mathfrak{A} - i\lambda)$ has closed range. In fact, suppose that $\{g_n\}_{n=1}^{\infty}$ is a sequence from the range of $(\mathcal{A} - i\lambda)$, and $\|g_n - g\|_p \to 0$. Then $(\mathcal{A} - i\lambda)J_\lambda g_n = g_n$. Since $J_\lambda g_n \to J_\lambda g$, and $(\mathcal{A} - i\lambda)$ is a closed operator, $(\mathcal{A} - i\lambda)J_\lambda g = g$.

Now suppose $\lambda \in \mathbf{R}$ is an accumulation point of $\mathcal{S}$. If $\lambda \neq 0$ (resp., if $\lambda = 0$) then by (1.8) we have for each ε such that $0 < \varepsilon < |\lambda|$ (resp., $0 < \varepsilon$) that there are $n_\varepsilon \in \mathbf{Z}^+$, $\eta_\varepsilon > 0$, and a set $Y_\varepsilon \subseteq C$ such that $\mu(Y_\varepsilon) > 0$ and $\eta_\varepsilon \leqslant |\lambda - n_\varepsilon \delta(z)| < \varepsilon$ for $z \in Y_\varepsilon$. Clearly $n_\varepsilon > 0$. Define f_ε in W^p by $f_\varepsilon(z, w) = \chi_{Y_\varepsilon}(z) w^{n_\varepsilon}$. It is easy to see that $(\mathcal{A} - i\lambda)f_\varepsilon = i(n_\varepsilon \delta(z) - \lambda)f_\varepsilon$ and so $\|(\mathcal{A} - i\lambda)f_\varepsilon\|_p \leqslant \varepsilon \|f_\varepsilon\|_p$. Let $g_\varepsilon = f_\varepsilon / \|f_\varepsilon\|_p$, whence

$$\|(\mathcal{A} - i\lambda)g_\varepsilon\|_p \leqslant \varepsilon. \tag{1.29}$$

Let $\mathcal{L}$ be the operator from the graph of $(\mathcal{A} - i\lambda)$ onto the range of $(\mathcal{A} - i\lambda)$ defined by $\mathcal{L}(f, (\mathcal{A} - i\lambda)f) = (\mathcal{A} - i\lambda)f$. If the range of $(\mathcal{A} - i\lambda)$ were closed, then by the open mapping theorem we would have that the $\mathcal{L}$-image of the closed ball $\mathcal{B}$ in the graph of $(\mathcal{A} - i\lambda)$ centered at 0 and of radius $[2(1 + \|Q_\lambda\|)]^{-1}$ would contain a closed ball around 0 in the range of $(\mathcal{A} - i\lambda)$. By Theorem (1.10)(i) and Theorem (1.15)(i), g_ε is in the range of $(\mathcal{A} - i\lambda)$. With the aid of (1.29) we get that for sufficiently small ε, there is $G_\varepsilon \in \mathfrak{D}(\mathcal{A})$ such that $\|G_\varepsilon\|_p \leqslant [2(1 + \|Q_\lambda\|)]^{-1}$ and $G_\varepsilon - g_\varepsilon$ is in the null-space of $(\mathcal{A} - i\lambda)$. Since g_ε is in the range of $(\mathcal{A} - i\lambda)$, $G_\varepsilon - Q_\lambda G_\varepsilon = g_\varepsilon$. This gives the absurd conclusion $1 = \|g_\varepsilon\|_p \leqslant (1 + \|Q_\lambda\|)\|G_\varepsilon\|_p \leqslant 1/2$.

We now establish Theorem (1.5) by showing that $\Lambda(\mathcal{A}) = i\mathcal{S}$.

PROOF OF THEOREM (1.5). By Theorem (1.15)(ii) (resp., (1.6)), $0 \in \Lambda(\mathcal{A})$ (resp., $0 \in \mathcal{S}$). Let $\lambda \in \mathbf{R}$, $\lambda \neq 0$. If $\lambda \in \mathcal{S}$, then λ is an accumulation point of $\mathcal{S}$, or else by (1.6) and (1.8) $\mu(E_{n,\lambda}) > 0$ for some positive integer n. In the first instance the range of $(\mathcal{A} - i\lambda)$ is not closed by Theorem (1.17), while in the second instance $(\mathcal{A} - i\lambda)$ is not one-to-one by Theorem (1.16). So $i\mathcal{S} \subseteq \Lambda(\mathcal{A})$. Conversely, if $i\lambda \in \Lambda(\mathcal{A})$, then the range of $(\mathcal{A} - i\lambda)$ is not closed, or else by Theorem (1.10)(iii) $(\mathcal{A} - i\lambda)$ is not one-to-one. In either event $\lambda \in \mathcal{S}$. This completes the proof.

REMARKS. Let $\lambda \in \mathbf{R}$. (i) If $\lambda \notin \mathcal{S}$, then the operator J_λ of Theorem (1.17) is $(\mathcal{A} - i\lambda)^{-1}$, the resolvent operator of $\mathcal{A}$ at $i\lambda$. (ii) If λ is not an accumulation point of $\mathcal{S}$, then it is easy to see with the aid of Theorem (1.10)(ii), Theorem (1.15)(ii), and the definition of J_λ in the proof of Theorem (1.17) that the null space of $(\mathcal{A} - i\lambda)$ coincides with the null space of J_λ. Hence $J_\lambda = J_\lambda(I - Q_\lambda)$. In particular, the range of J_λ is the range of its restriction to the range of $(\mathcal{A} - i\lambda)$. Thus if λ is not an accumulation point of $\mathcal{S}$ and $(\mathcal{A} - i\lambda)$ is not one-to-one, then the range of J_λ is not $\mathfrak{D}(\mathcal{A})$. For example, if $\delta = \chi_Y$, where $0 < \mu(Y) < 1$, then $\mathcal{S} = \mathbf{Z}^+$, and $(\mathcal{A} - i)$ is not one-to-one.

We close this section with a discussion of the problem of extending $\{S_t\}$, $t \in \mathbf{R}$, to a semigroup of contraction operators indexed by the upper half-plane in $\mathbf{C}$. Specifically, let $\mathcal{H} = \{\zeta \in \mathbf{C}\colon \operatorname{Im} \zeta \geqslant 0\}$. By the term "contraction operator" we shall mean an operator of norm at most one which maps $H^p(C^2)$ into itself. Suppose the mapping $t \in \mathbf{R} \mapsto S_t$ can be extended to a mapping $\zeta \in \mathcal{H} \mapsto S_\zeta$ so that $\{S_\zeta\}$, $\zeta \in \mathcal{H}$, is a semigroup of contraction operators which is continuous on $\mathcal{H}$ in the strong operator topology and holomorphic on the interior of $\mathcal{H}$. Let $\mathcal{G}$ be the infinitesimal generator of $\{S_{iy}\}$, $y \geqslant 0$. By [7, Theorem 17.9.2], $\mathcal{G} = i\mathcal{Q}$. By [4, VIII. 1.11], $\mathrm{Re}[\Lambda(\mathcal{G})] \leqslant 0$. So $\mathrm{Im}[\Lambda(\mathcal{Q})] \geqslant 0$, and, by Theorem (1.5), $\mathcal{E} \geqslant 0$. It follows that $\delta(z) \geqslant 0$ for almost all z in C. The next result shows that conversely if $\delta \geqslant 0$ a.e., then $\{S_t\}$, $t \in \mathbf{R}$, can be extended to a semigroup $\{S_\zeta\}$, $\zeta \in \mathcal{H}$, as above. Without loss of generality we take $\delta \geqslant 0$ everywhere on C.

(1.30) THEOREM. *Suppose $\delta(z) \geqslant 0$ for all $z \in C$. Then the one-parameter group $\{S_t\}$ in Theorem* (1.10) *has a unique extension to a semigroup $\{S_\zeta\}$, $\zeta \in \mathcal{H}$, of bounded operators on $H^p(C^2)$ such that $\{S_\zeta\}$ is continuous on $\mathcal{H}$ in the strong operator topology and holomorphic on the interior of $\mathcal{H}$. The unique semigroup $\{S_\zeta\}$, $\zeta \in \mathcal{H}$, consists of contraction operators.*

PROOF. Uniqueness is immediate from the Schwarz reflection principle.

Suppose $\mathrm{Im}\,\zeta > 0$ and $f \in H^p(C^2)$. For almost all $z \in C, f(z, \cdot) \in H^p(d\psi/2\pi)$. For such z, let $\tilde{f}(z, w)$ denote (for $|w| < 1$) the Poisson integral of $f(z, \cdot)$. We define h_ζ almost everywhere on C^2 as follows: $h_\zeta(z, w) = \tilde{f}(z, e^{i\delta(z)\zeta}w)$ if $\delta(z) > 0$, and $h_\zeta(z, w) = f(z, w)$ if $\delta(z) = 0$. It is easy to see that $h_\zeta \in H^p(C^2)$, and $\|h_\zeta\|_p \leqslant \|f\|_p$. Let $S_\zeta f = h_\zeta$. Then S_ζ is a contraction operator. For $\zeta \in \mathcal{H}$, it is easy to see that if $f \in H^p(C^2)$, then for almost all $z \in C$, $(S_\zeta f)(z, w)$ has (as a function of w) the sequence of Fourier coefficients $\{e^{in\delta(z)\zeta}a_f^{(n)}(z)\}$. It follows readily that $S_{\zeta+\eta} = S_\zeta S_\eta$ for $\zeta, \eta \in \mathcal{H}$. To complete the proof it suffices to show that if $f_0 \in H^p(C^2)$, q is the index conjugate to p, and $g \in L^q(\tilde{\mu})$, then, as a function of ζ,

$$F(\zeta) \equiv \int_{C^2} g(z, w)[(S_\zeta f_0)(z, w)]\, d\tilde{\mu}(z, w)$$

is continuous on $\mathcal{H}$ and analytic on the interior of $\mathcal{H}$. Since $S_\zeta u = u$ for $u \in Z^p$, $\zeta \in \mathcal{H}$, we can assume without loss of generality that $f_0 \in W^p$. So there is a sequence $\{v_k\}$ from the linear span of the functions $z^m w^n$ ($m, n \in \mathbf{Z}, n > 0$) such that $v_k \to f_0$ in $L^p(\tilde{\mu})$. Since $\|S_\zeta\| \leqslant 1$ for $\zeta \in \mathcal{H}$, the functions (of ζ)

$$\int_{C^2} g(z, w)[(S_\zeta v_k)(z, w)]\, d\tilde{\mu}(z, w)$$

tend to F uniformly on $\mathcal{H}$ as $k \to \infty$. Accordingly we can further assume without loss of generality that $f_0(z, w) \equiv z^m w^n$ for some $m \in \mathbf{Z}$, $n \in \mathbf{Z}$, $n > 0$. Hence $(S_\zeta f_0)(z, w) = e^{in\delta(z)\zeta} z^m w^n$ for $\zeta \in \mathcal{H}$. The continuity of F on $\mathcal{H}$ now follows from the dominated convergence theorem.

Let ζ_0 be an interior point of $\mathcal{K}$, $y_0 = \operatorname{Im} \zeta_0$. Let $\{\zeta_k\}$ be a sequence of points distinct from ζ_0 such that $|\zeta_k - \zeta_0| < y_0/2$ for all k and $\zeta_k \to \zeta_0$. For all k

$$
\begin{aligned}
(1.31)\quad & [F(\zeta_k) - F(\zeta_0)](\zeta_k - \zeta_0)^{-1} \\
& = \int_{C^2} g(z,w) z^m w^n \left(e^{in\delta(z)\zeta_k} - e^{in\delta(z)\zeta_0}\right)(\zeta_k - \zeta_0)^{-1} \, d\tilde{\mu}(z,w).
\end{aligned}
$$

But for each k and all $z \in C$

$$|e^{in\delta(z)\zeta_k} - e^{in\delta(z)\zeta_0}|/|\zeta_k - \zeta_0| \leqslant n\delta(z) e^{-n\delta(z)y_0/2}.$$

Since the majorant in this last inequality is a bounded function of z, it follows from (1.31) by dominated convergence that

$$F'(\zeta_0) = in \int_{C^2} \delta(z) e^{in\delta(z)\zeta_0} z^m w^n g(z,w) \, d\tilde{\mu}(z,w).$$

2. **Spectral analysis when $\{T_t|Z^p\}$ has an unbounded generator.** For $1 \leqslant p < \infty$, let Ω_p be the set of all strongly continuous one-parameter groups $\{T_t\}$ of isometries of $H^p(C^2)$ such that $\{T_t|Z^p\}$ is not continuous in the uniform operator topology (if $p = 2$, we further require that each T_t have the form (1.1)). It is known [3, Theorem (2.22) and (2.24)] that a group $\{T_t\} \in \Omega_p$, $1 \leqslant p < \infty$, can be written in the form (0.2). The unique group of Möbius transformations of the disc $\{\phi_t\}$ occurring in (0.2) is called the *conformal group* of $\{T_t\}$. As described in the Introduction, the conformal group of $\{T_t\}$ is assigned a type number depending on the nature of its set of common fixed points in the extended plane. $\{T_t\}$ is said to be of type (i), (ii), or (iii) according as its conformal group is.

(2.1) THEOREM. *Let $\{T_t\} \in \Omega_p$, $1 \leqslant p < \infty$. If $\{T_t\}$ is of type* (ii) *or* (iii), *then the spectrum of its infinitesimal generator is* $i\mathbf{R}$.

PROOF. In the case at hand, $\{T_t\}$ can be expressed in the form (0.2) with the real constant δ equal to zero [3, Theorem (2.24)]. There is clearly no loss of generality if we establish the desired conclusion under the additional hypothesis that the real constant ρ in (0.2) is zero. Let U be the isometry of $H^p(C^2)$ onto itself defined by $(Uf)(z,w) = f(z, u(z)w)$, where u is the unimodular measurable function on C in (0.2). Since it suffices to establish the desired conclusion for the group $\{UT_tU^{-1}\}$, we assume further without loss of generality that the function u in (0.2) is identically one.

If $\{T_t\}$ is of type (ii) (resp., type (iii)), we represent its conformal group $\{\phi_t\}$ in the form [1, (1.8)] (resp., [1, (1.9)]). This representation furnishes a certain nonzero real constant c, and a unimodular common fixed point α of the group

$\{\phi_t\}$. Let λ be any real number. For each $\eta > -p^{-1}$ (resp., for each $\eta \in \mathbf{C}$ with $\operatorname{Re}(\eta) > -p^{-1}$) define f_η on $C\backslash\{\alpha\}$ by setting

$$f_\eta(z) = (z-\alpha)^\eta \exp(\lambda\alpha/c(z-\alpha)) \quad (\text{resp., } f_\eta(z) = (z-\alpha)^\eta).$$

Define $F_\eta(z,w) = f_\eta(z)w$ for $z \in C\backslash\{\alpha\}$, $w \in C$. It is straightforward to verify that $F_\eta \in W^p$. Denote the infinitesimal generator of $\{T_t\}$ by $\mathscr{Q}$. By virtue of the description of $\mathscr{Q}$ in [3, (3.16) and Theorem (3.17)], the proof of [1, Theorem (3.1)] applies with obvious minor modifications to show that $\|(\mathscr{Q} + i\lambda)F_\eta\|_p/\|F_\eta\|_p$ (resp., $\|(\mathscr{Q} - i\lambda)F_\eta\|_p/\|F_\eta\|_p$) tends to zero as η approaches $-p^{-1}$ from the right (resp., as η approaches $(-p^{-1} + i\lambda c^{-1})$).

Remark. In the type (ii) part of the proof of Theorem (2.1) above, the function $f_\eta \in L^p(\mu)$ (and hence $F_\eta \in W^p$), but f_η need not be in $H^p(d\theta/2\pi)$. This state of affairs accounts for the fact that the infinitesimal generator of a type (ii) group on $H^p(d\theta/2\pi)$ has for its spectrum a proper subset of $i\mathbf{R}$ [1, Theorem (3.1)(ii)] in contrast to the outcome of Theorem (2.1) for the torus.

Now let $\{T_t\} \in \Omega_p$ be of type (i) with the representation (0.2). Let c be the angular velocity of $\{\phi_t\}$ (see [2, p. 337]; in particular $c \in \mathbf{R}$, $c \neq 0$). As shown in [3, (2.26) and Theorem (3.1)] the cohomology class of the cocycle of $\{T_t\}$ can be identified with the element $\delta + c\mathbf{Z}$ of the group $\mathbf{R}/c\mathbf{Z}$. The order of this cohomology class will be called the *cohomological order* of $\{T_t\}$. Thus the cohomological order of $\{T_t\}$ is finite (resp., infinite) if and only if δ/c is rational (resp., irrational).

(2.2) Theorem. *Let $\{T_t\} \in \Omega_p$ $(1 \leqslant p < \infty)$ be of type* (i), *with infinitesimal generator A. Let $\{T_t\}$ have the form* (0.2), *and let c be the angular velocity of its conformal group. If the cohomological order of $\{T_t\}$ is a positive integer k, then A has pure point spectrum, and $\Lambda(A) = i(\rho + cp^{-1} + ck^{-1}\mathbf{Z})$. If the cohomological order of $\{T_t\}$ is infinite, then $\Lambda(A) = i\mathbf{R}$.*

Proof. For $f \in L^p(\tilde{\mu})$, let $\{\hat{f}_{m,n}\}$ be the double sequence of Fourier coefficients of f. Let δ_0 be the unique element of $\delta + c\mathbf{Z}$ such that $0 < \delta_0 \leqslant |c|$. By [3, Theorem (4.4)] A is similar under an isometry to $\mathscr{Q} + i(\rho + cp^{-1})$, where $\mathscr{Q}$, the infinitesimal generator of the group $\{S_t\}$ given by

$$(S_t f)(z,w) = f(e^{ict}z, e^{i\delta_0 t}w), \quad \text{for } f \in H^p(C^2),$$

has the following description: $\mathscr{Q}f = g$ means $\{\hat{g}_{m,n}\} = \{i(mc + n\delta_0)\hat{f}_{m,n}\}$. In particular, for each $(m,n) \in P$ (recall the definition of P in the Introduction), $z^m w^n$ is an eigenvector of $\mathscr{Q}$ with associated eigenvalue $i(mc + n\delta_0)$.

If the cohomological order of $\{T_t\}$ is a positive integer k, then $\delta_0|c|^{-1} = jk^{-1}$, where j is a positive integer relatively prime to k. Because j and k are relatively prime, any integer n can be written in the form $n = n_1 j + n_2 k$, where $n_1 > 0$. It follows that $\{mc + n\delta_0 : (m,n) \in P\} = \{mc + n\delta_0 : m \in \mathbf{Z}, n \in \mathbf{Z}\}$. Also, since j and k are relatively prime, we have $\{mc + n\delta_0 : m \in \mathbf{Z}, n \in \mathbf{Z}\} = ck^{-1}\mathbf{Z}$. If $\lambda \in \mathbf{R}\backslash(ck^{-1}\mathbf{Z})$, let $\lambda_1 = \lambda c^{-1}$, $\delta_1 = \delta_0 c^{-1}$, and define a Borel measure β on C^2 as follows: for each continuous complex-valued function f on C^2

$$\beta(f) = -c^{-1}(e^{i2\pi k\lambda_1} - 1)^{-1} \int_0^{2\pi k} f(e^{i\theta}, e^{i\delta_1\theta}) e^{i\lambda_1\theta}\, d\theta.$$

Denoting the double sequence of Fourier coefficients of the measure β by $\{\hat{\beta}_{m,n}\}$ we have for all $m \in \mathbf{Z}$, $n \in \mathbf{Z}$,

$$\hat{\beta}_{m,n} = i(\lambda - mc - n\delta_0)^{-1}. \tag{2.3}$$

For $g \in H^p(C^2)$, let F be the convolution on C^2, $g * \beta$. Then for $m \in \mathbf{Z}$, $n \in \mathbf{Z}$, $\hat{F}_{m,n} = i(\lambda - mc - n\delta_0)^{-1}\hat{g}_{m,n}$. Clearly, $F \in H^p(C^2)$ and $(\mathcal{Q} - i\lambda)F = g$. It is obvious from double Fourier coefficients that $(\mathcal{Q} - i\lambda)$ is one-to-one, and thus we have established that $\mathcal{Q}$ has pure point spectrum equal to $ick^{-1}\mathbf{Z}$.

If the cohomological order of $\{T_t\}$ is infinite, then $\delta_0 c^{-1}$ is irrational, and therefore $\{i(mc + n\delta_0) : (m,n) \in P\}$ is dense in $i\mathbf{R}$. So $\Lambda(\mathcal{Q}) = i\mathbf{R}$.

Remark. By direct calculation using (2.3), it is easy to see that for each $s \in \mathbf{Z}$ the Fourier coefficient of β corresponding to the integer pair $((-\operatorname{sgn} c)sj, sk)$ is i/λ. So by Riemann-Lebesgue β is not absolutely continuous with respect to $\tilde{\mu}$.

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